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Perturbations of Moore–Penrose Metric Generalized Inverses of Linear Operators in Banach Spaces

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Abstract In this paper, the perturbations of the Moore–Penrose metric generalized inverses of linear operators in Banach spaces are described. The Moore–Penrose metric generalized inverse is homogeneous and nonlinear in general, and the proofs of our results are different from linear generalized inverses. By using the quasi-additivity of Moore–Penrose metric generalized inverse and the theorem of generalized orthogonal decomposition, we show some error estimates of perturbations for the single-valued Moore–Penrose metric generalized inverses of bounded linear operators. Furthermore, by means of the continuity of the metric projection operator and the quasi-additivity of Moore–Penrose metric generalized inverse, an expression for Moore–Penrose metric generalized inverse is given.

Keywords Banach space, Moore–Penrose metric generalized inverse, perturbation

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1 Introduction

The concept of generalized inverses has been extensively studied in the last decades, which has its genetic in the context of the so-called "ill-posed" linear problems. There has been a great deal of interest in the theory and applications of generalized inverses (see [3, 11–14, 18], etc.), however, linearly generalized inverses are not suitable to construct the extremal solutions, the minimal norm solutions, and the best approximation solutions of an ill-posed linear operator equations in Banach spaces [15]. In order to solve the best approximation problems for ill-posed linear operator equations in Banach spaces, it is necessary to study the metric generalized

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inverses of linear operators between Banach spaces. This kind of generalized inverses, which are set-valued bounded homogeneous operators, was introduced by Nashed and Votruba in 1974 in [15]. Moreover, in 2003, H. Wang and Y. W. Wang introduced the notion of Moore–Penrose metric generalized inverses of linear operators between Banach spaces [22], which are not only generalized inverses but also bounded homogeneous and nonlinear (in general) operators.

Throughout this paper, perturbation theory means the perturbation theory for linear operators, which was created by Rayleigh and Schrödinger [9], and it occupies an important place in applied mathematics. In the last years the group of mathematicians working in the perturbation theory, involved several directions in analytical dynamics and nonlinear oscillation theory. During the last decades it has grown into a mathematical discipline with its own interests and techniques. There is a wide literature of the results towards the perturbation for linear operators, especially linear generalized inverses [4, 7, 21, 23–25], etc. Although the perturbation of linear generalized inverses of operators have been widely studied, and numerous results were obtained, the problems of nonlinear generalized inverses remaind unsolved except some initiated study in [10, 17].

In 1997, Chen and Xue extended some results in the perturbation analysis of bounded linear operators in Banach spaces to a more general situation (see [4]). By using this result, they give some results of perturbation analysis for the operator equation $Tx = b$ through linearly generalized inverse T^+ (see [4]). In 2006, some descriptions concerning the solution of the equality $Tx = b$ through the Moore–Penrose metric generalized inverse were obtained in [10] by us. The starting point and initial motivation for [10] was the result of [4].

In Section 3, the perturbations of Moore–Penrose metric generalized inverses for operators between Banach spaces will be further studied. By using the generalized orthogonal decomposition theorem (Theorem 2.6) and the quasi-additivity of Moore–Penrose metric generalized inverses (Theorem 3.1), we obtain a description of Moore–Penrose single-valued metric generalized inverses of operators on Banach spaces.

2 Preliminaries

Throughout this paper, let X and Y be two real Banach spaces, $B(X,Y)$ be the space of all bounded linear operators from X to Y, $H(X,Y)$ be the space of all bounded homogenous operators from X to Y, $H(X,X) := H(X)$, and $L(X)$ be the space of all linear operators from X to X. Denote by $D(T)$, $R(T)$ and $N(T)$ the domain, the range and the null space of T, respectively. Let L be a subspace of X, $L^\perp := \{x^* \in X^* : \langle x^*, x\rangle = 0, x \in L\}$.

Definition 2.1 ([2]) *Let X^* be the dual space of X. The set-valued mapping $F_X : X \to 2^{X^*}$ defined by*

$$F_X(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2,\ x \in X\}$$

is called the duality mapping of X, where $\langle x^*, x\rangle$ denotes the value of x^* at the point x.

Definition 2.2 ([6]) *If $K \subset X$, the set-valued mapping $\mathcal{P}_K : X \to K$ defined by*

$$\mathcal{P}_K(x) = \Big\{y \in K : \|x-y\| = \inf_{y\in K} \|x-y\|\Big\},\quad x \in X,$$

is called the metric projection.

The kernel of the metric projection $\mathcal{P}_K$ onto a proximinal subspace K is the set

$$\ker \mathcal{P}_K := \{x \in X : 0 \in \mathcal{P}_K(x)\} = \Big\{x \in X \big| \|x\| = \inf_{y\in K} \|x-y\|\Big\}.$$

Remark 2.3 ([18]) If $K \subset X$, then K is said to be proximinal if $\mathcal{P}_K(x) \neq \emptyset$ for any $x \in X$. K is said to be semi-Chebyshev if $\mathcal{P}_K(x)$ is at most a single point set for each $x \in X$. K is called a Chebyshev set if it is both proximinal and semi-Chebyshev. When K is a Chebyshev set, we will denote $\mathcal{P}_K(x)$ by $\pi_K(x)$, furthermore, $\pi_K(x)$ satisfies

(1) $\pi_K^2(x) = \pi_K(x), \forall\, x \in X$;
(2) $\pi_K(\lambda x) = \lambda \pi_K(x), \forall\, x \in X, \lambda \in \mathbb{R}$;
(3) $\pi_K(x+y) = \pi_K(x) + y, \forall\, x \in X, y \in K$;
(4) $\|\pi_K(x)\| \leq 2\|x\|$.

Definition 2.4 ([1]) *Let U and V be linear spaces, and $S \subset U$ be a subset of U. A mapping $T : U \to V$ is called quasi-additivity on S, if T satisfies*

$$T(x+y) \;=\; T(x) \;+\; T(y), \quad x \in U,\; y \in S.$$

Definition 2.5 ([22]) *Let $T : D(T) \subset X \to Y$ be a Linear operator, $\overline{N(T)}$ and $\overline{R(T)}$ be Chebyshev subspaces of X and Y, respectively. If there exists a homogeneous operator T^M: $D(T^M) \to D(T)$ such that*

(1) *$TT^MT = T$ on $D(T)$;*
(2) *$T^MTT^M = T^M$ on $D(T^M)$;*
(3) *$T^MT = I_{D(T)} - \pi_{\overline{N(T)}}$ on $D(T)$;*
(4) *$TT^M = \pi_{\overline{R(T)}}$ on $D(T^M)$,*

then T^M is called the Moore–Penrose metric generalized inverse of T, where $I_{D(T)}$ is the identity operator on $D(T)$, $D(T^M) = R(T) \dotplus F_Y^{-1}(R(T)^\perp)$ and F_Y is the dual mapping of Y.

Theorem 2.6 ([20] Generalized orthogonal decomposition theorem) *Let L be a proximinal subspace of X. Then for any $x \in X$, we have the decomposition*

$$x = x_1 + x_2,$$

where $x_1 \in L$ and $x_2 \in F_X^{-1}\left(L^\perp\right)$. In this case we have $X = L + F_X^{-1}(L^\perp)$. If L is a Chebyshev subspace of X, then the decomposition is unique and

$$x = \mathcal{P}_L(x) + x_2, \quad x_2 \in F_X^{-1}(L^\perp).$$

In this case we have $X = \mathcal{P}_L(x) \dotplus F_X^{-1}(L^\perp)$, where $\mathcal{P}_L(x) = \{\pi_L x\}$.

Lemma 2.7 ([6]) *Let K be a proximinal subspace of the normed linear space E. Then $\ker \mathcal{P}_K$ is a subspace if and only if K is Chebyshev and $\mathcal{P}_K$ is linear.*

Lemma 2.8 ([19]) *If $T \in H(X,Y)$, the addition and the scalar multiplication are defined as usual in linear structures. If the norm of T is defined as*

$$\|T\| = \sup_{\|x\|=1} \|Tx\|, \quad T \in H(X,Y), \tag{2.1}$$

then $(H(X,Y), \|\cdot\|)$ is a Banach space.

Definition 2.9 ([8]) *A nonempty subset C of X is said to be approximately compact, if for any sequence $\{x_n\}$ in C and any $y \in X$ such that* $\|x_n - y\| \to \operatorname{dist}(y, C) := \inf\{\|y-z\| : z \in C\}$, *we have that* $\{x_n\}$ *has a Cauchy subsequence.* X *is called approximately compact if any nonempty closed and convex subset of* X *is approximately compact.*

Lemma 2.10 ([5]) *Let* C *be a semi-Chebyshev closed subset of* X*. If* C *is approximatively compact, then* C *is a Chebyshev subset of* X*, and the metric projector* π_C *is continuous.*

Lemma 2.11 *Let* $T : D(T) \subset X \to Y$ *be a Linear operator. If* T *has a Moore–Penrose metric generalized inverse* T^M*, then*

(1) T^M *is unique on* $D(T^M)$*, and* $T^M y = (T|_{C(T)})^{-1}\pi_{\overline{R(T)}} y$ *when* $y \in D(T^M)$*, where* $D(T^M) = R(T) \dotplus F_Y^{-1}(R(T)^{\perp})$*,* $C(T) = D(T) \cap F_X^{-1}(N(T)^{\perp})$*;*

(2) *there exists a linear inner inverse* T^- *from* $R(T)$ *to* $D(T)$ *(i.e.,* $TT^-T = T$*) such that*

$$T^M y = (I_{D(T)} - \pi_{\overline{N(T)}})T^- \pi_{\overline{R(T)}} y \tag{2.2}$$

for $y \in D(T^M)$*.*

Remark 2.12 This result has been obtained in [22] by H. Wang and Y. W. Wang under the assumption that the underlying Banach space X and Y are strictly convex, but it is easy to show that the result remains valid under the weaker assumption that $N(T)$ and $R(T)$ are Chebyshev subspaces of X and Y, respectively.

Theorem 2.13 ([16, 18]) *Let* $T \in B(X, Y)$*,* $\overline{N(T)}$ *and* $\overline{R(T)}$ *be Chebyshev subspaces of* X *and* Y*, respectively. Then there exists a unique Moore–Penrose metric generalized inverse* T^M *of* T *such that*

$$T^M(y) = (T\,|_{C(T)})^{-1}\pi_{\overline{R(T)}}(y)$$

for any $y \in D(T^M)$*, where* $D(T^M) = R(T) \dotplus F_Y^{-1}(R(T)^{\perp})$*,* $C(T) = D(T) \cap F_X^{-1}(N(T)^{\perp})$*.*

Remark 2.14 In Theorem 2.13, if $\pi_{\overline{R(T)}}$ and $(T|_{C(T)})^{-1}$ are all bounded homogenous operators, then T^M is also bounded homogenous operator. Thus, the norm of T^M is well defined by (2.1) in Lemma 2.8.

In 2006, the following perturbation result about the solution of the equality $Tx = b$ through the theory of Moore–Penrose metric generalized inverse is given in [10].

Theorem 2.15 ([10]) *Let* T *and* $\delta T \in B(X, Y)$*,* $N(T)$ *and* $\overline{R(T)}$ *be Chebyshev subspaces of* X *and* Y*, respectively. If* $\overline{T} = T + \delta T$*,* $b \in R(T)$ *and* $b \neq 0$*, then for every* $\overline{x} \in S(\overline{T}, b)$*, we have*

$$\|T\|^{-1}\|\delta T\overline{x}\| \leq \operatorname{dist}(\overline{x}, S(T, b)) \leq \|T^M\|\|\delta T\|\|\overline{x}\|,$$

where $S(T, b) = \{x \in X : Tx = b\ \}, S(\overline{T}, b) = \{\overline{x} \in X : \overline{T}\overline{x} = b\}$*.*

When T has linearly generalized inverse T^+, Theorem 2.15 is Lemma 4.1 of [4].

3 Main Results

In general, the metric generalized inverse of operator is a bounded homogeneous nonlinear operator, which suggests that the discussion will be different from the perturbation of linear generalized inverse. At first, we discuss the quasi-additivity of T^M.

Theorem 3.1 *Let* $T \in B(X, Y)$*. If* $N(T)$ *is a proximinal subspace of* X*,* $R(T)$ *is a Chebyshev subspace of* Y *and* $\ker \mathcal{P}_{N(T)}$ *is a subspace of* X*, then*

(1) *there exists a unique Moore–Penrose metric generalized inverse* T^M *of* T*, and*

$$T^M y = (I_{D(T)} - \pi_{N(T)})T^- \pi_{R(T)} y, \quad \forall y \in Y, \tag{3.1}$$

where T^- *is a linear inner inverses of* T;

(2) T^M *is quasi-additive (i.e,* T^M *is quasi-additive on* $R(T)$*),*

$$T^M(x+y) = T^M x + T^M y$$

for all $x \in Y$, $y \in R(T)$.

Proof (1) Since $N(T)$ is a proximinal subspace of X and $\ker \mathcal{P}_{N(T)}$ is a subspace of X, it follows from Lemma 2.7 that $N(T)$ is a Chebyshev subspaces of X. Since $N(T)$ and $R(T)$ are Chebyshev subspaces of X and Y, respectively, by Lemma 2.11 and Theorem 2.13, there exists a unique Moore–Penrose meric generalized inverse T^M of T such that

$$T^M y = (I_{D(T)} - \pi_{N(T)})T^- \pi_{\overline{R(T)}} y, \quad \forall y \in D(T^M),$$

where $D(T^M) = R(T) \dotplus F_Y^{-1}(R(T)^{\perp})$, and T^- is a linear inner inverses of T. Since $R(T)$ is a Chebyshev subspace of Y, then $D(T^M) = Y$ by Theorem 2.6. Therefore, (3.1) is valid.

(2) Noticing that $N(T)$ is a proximinal subspace of X and $\ker \mathcal{P}_{N(T)}$ is a linear subspace of X, Lemma 2.7 implies that $\pi_{N(T)}$ is a linear operator. Thus $I_{D(T)} - \pi_{N(T)}$ is a linear operator. By Lemma 2.11, there exists a linear inner inverse T^- of T. Moreover, $\pi_{\overline{R(T)}} = \pi_{R(T)}$ is bounded quasi-linear (quasi-additive) homogeneous metric projector, which shows that T^M is a bounded homogeneous operator. Thus for each $x \in Y$, $y \in R(T)$, we have

$$\begin{aligned}
T^M(x+y) &= (I_{D(T)} - \pi_{N(T)})T^- \pi_{R(T)}(x+y) \\
&= (I_{D(T)} - \pi_{N(T)})T^- [\pi_{R(T)} x + y] \\
&= (I_{D(T)} - \pi_{N(T)})T^- \pi_{R(T)} x + (I_{D(T)} - \pi_{N(T)})T^- y \\
&= T^M x + (I_{D(T)} - \pi_{N(T)})T^- y \\
&= T^M x + T^M y,
\end{aligned}$$

which finishes the proof. □

Corollary 3.2 *Let* $T \in B(X,Y)$ *and* $\delta T \in B(X,Y)$. $N(T)$ *is a proximinal subspace of* X*,* $R(T)$ *is a Chebyshev subspace of* Y*. If we assume that* $\ker \mathcal{P}_{N(T)}$ *is a subspace of* X *and* $R(\delta T) \subset R(T)$*, then* $T^M \delta T$ *is a linear operator.*

Proof By Theorem 3.1, there exists a unique Moore–Penrose metric generalized inverse T^M of T such that

$$T^M y = (I_{D(T)} - \pi_{N(T)})T^- \pi_{R(T)} y, \quad \forall y \in Y.$$

By $R(\delta T) \subset R(T)$, it is easy to see that

$$T^M \delta T = (I_{D(T)} - \pi_{N(T)})T^- \pi_{R(T)}\, \delta T = (I_{D(T)} - \pi_{N(T)})T^- \delta T.$$

Therefore, $T^M \delta T$ is also a linear operator because $(I_{D(T)} - \pi_{N(T)})T^- \delta T$ is a linear operator. The proof is complete. □

In order to prove Theorem 3.5, we need the following results.

Lemma 3.3 *Let* $T \in H(X)$*. If* T *is quasi-additive on* $R(T)$ *and* $\|T\| < 1$*, then the operator* $(I-T)^{-1}$ *exists and*

(1) $(I-T)^{-1} \in H(X)$;

(2) $(I-T)^{-1} = \sum_{k=0}^{\infty} T^k$;

(3) $\|(I-T)^{-1}\| \le \frac{1}{1-\|T\|}$;
(4) $\|(I-T)^{-1}-I\| \le \frac{\|T\|}{1-\|T\|}$.

Proof Let $A_n = \sum_{k=0}^{n} T^k$ for all nonnegative integers n. Then A_n are bounded homogenous operators. For all $n > m$, we have

$$\|A_n - A_m\| = \left\| \sum_{k=m}^{n} T^k \right\| \le \sum_{k=m}^{n} \|T\|^k \to 0$$

as $m, n \to \infty$. By the completeness of $H(X)$, there exists a unique operator $A \in H(X)$ such that

$$A = \lim_{n\to\infty} A_n = \sum_{k=0}^{\infty} T^k.$$

Since T is quasi-additive on $R(T)$, we have

$$T(I + T + T^2 + \cdots + T^n) = T + T^2 + \cdots + T^{n+1}.$$

Hence,

$$(I-T)A_n = (I-T)(I + T + T^2 + \cdots + T^n) = I - T^{n+1}$$

and

$$A_n(I-T) = I - T^{n+1}$$

for each $n \ge 1$. Letting $n \to \infty$, we obtain $A = (I-T)^{-1}$. Therefore,

$$\|(I-T)^{-1}\| = \|A\| \le \frac{1}{1-\|T\|},$$

$$\|(I-T)^{-1} - I\| = \|A - I\| \le \frac{\|T\|}{1-\|T\|}.$$

This finishes the proof. □

Lemma 3.4 *Let $T \in B(X,Y)$, $\delta T \in B(X,Y)$ and $\overline{T} = T + \delta T$. Assume that $\ker \mathcal{P}_{N(T)}$ is a subspace of X, $N(T)$ is a proximinal subspace of X, $R(T)$ is a Chebyshev subspace of Y. If $\|T^M\|\|\delta T\| < 1$, $R(\delta T) \subset R(T)$ and $N(T) \subset N(\delta T)$, then*

$$R(T) = R(\overline{T}), \quad N(T) = N(\overline{T}).$$

Proof By Theorem 2.13, there exists a unique Moore–Penrose Metric Generalized inverse T^M of T, which is a bounded homogenous operator (see Remark 2.14). Since $TT^M = \pi_{R(T)}$, we have

$$\overline{T} = T + \delta T = T(I + T^M \delta T).$$

By the assumption that $\ker \mathcal{P}_{N(T)}$ is a subspace of X, it follows from Theorem 3.1 that T^M is quasi-additive on $R(T) \subset Y$. Moreover, $R(\delta T) \subset R(T)$, and therefore $T^M \delta T$ is quasi-additive on $R(T^M \delta T)$. Noticing that

$$\|T^M \delta T\| \le \|T^M\|\|\delta T\| < 1 \quad \text{and} \quad -T^M \delta T \in H(X),$$

by Lemma 3.3, the operator $(I - (-T^M \delta T))^{-1}$ exists and $(I + T^M \delta T)^{-1} \in H(X)$. Hence,

$$T = \overline{T}(I + T^M \delta T)^{-1},$$

which means that $R(T) \subset R(\overline{T})$. It is obvious that $R(\overline{T}) \subset R(T)$. Therefore,

$$R(T) = R(\overline{T}).$$

By the assumption that $N(T) \subset N(\delta T)$ we easily deduce that $N(T) \subset N(\overline{T})$. Noticing that

$$\|\delta T T^M\| \leq \|T^M\|\|\delta T\| < 1 \quad \text{and} \quad -\delta T T^M \in H(X),$$

by Lemma 3.3, the operator $(I - (-\delta T T^M))^{-1}$ exists and $(I + \delta T T^M)^{-1} \in H(X)$. By $T^M T = I - \pi_{N(T)}$, we get

$$\overline{T} = T + \delta T = (I + \delta T T^M)T.$$

Hence

$$T = (I + \delta T T^M)^{-1}\overline{T}.$$

On the other hand, $(I + \delta T T^M)^{-1}$ is a homogenous operator, so for any $x \in N(\overline{T})$, we have

$$Tx = (I + \delta T T^M)^{-1}\overline{T}x = 0,$$

which means that $x \in N(T)$. Therefore,

$$N(T) = N(\overline{T}).$$

This finishes the proof. □

Now, we are ready to state our result concerning the perturbation of Moore–Penrose metric generalized inverse T^M of T.

Theorem 3.5 *Let $T \in B(X,Y)$, $\delta T \in B(X,Y)$ and $\overline{T} = T + \delta T$. Assume that $N(T)$ is a proximinal subspace of X, $R(T)$ is a Chebyshev subspace of Y. If $\|T^M\|\|\delta T\| < 1$, $R(\delta T) \subset R(T)$, $N(T) \subset N(\delta T)$, and $\ker \mathcal{P}_{N(T)}$ is a subspace of X, then T^M and $\overline{T}^M$ exist. Moreover, we have*

$$\frac{\|\overline{T}^M - T^M\|}{\|\overline{T}^M\|} \leq \|T^M\|\|\delta T\| \leq \frac{\|T^M\|\|\delta T\|}{1 - \|T^M\|\|\delta T\|},$$

$$\|\overline{T}^M\| \leq \frac{\|T^M\|}{1 - \|T^M\|\|\delta T\|},$$

where $\|T^M\|$ is the norm of bounded homogenous operator for T^M.

Proof It follows from Lemma 3.4 that $R(T) = R(\overline{T})$, $N(T) = N(\overline{T})$, and so by Theorem 2.13, T^M and $\overline{T}^M$ exist and

$$D(T^M) = R(T) \dotplus F_Y^{-1}(R(T)^\perp), \quad D(\overline{T}^M) = R(\overline{T}) \dotplus F_Y^{-1}(R(\overline{T})^\perp),$$

where $F_Y : Y \to Y^*$ is the duality mapping of Y .

Since $R(T)$ and $R(\overline{T})$ are Chebyshev subspaces of Y, by Theorem 2.6,

$$D(\overline{T}^M) = D(T^M) = Y.$$

Since $R(\overline{T}) = R(T)$, for all $b \in R(\overline{T}) = R(T)$ and $b \neq 0$, $x = T^M b \in S(T,b) = \{x \in X : Tx = b\}$, $\overline{x} = \overline{T}^M b \in S(\overline{T},b) = \{\overline{x} \in X : \overline{T}\overline{x} = b\}$. Theorem 2.15 implies that

$$\operatorname{dist}(\overline{x}, S(T,b)) \leq \|T^M\|\|\delta T\|\|\overline{x}\|. \tag{3.2}$$

Noticing that

$$S(T,b) = T^M b + N(T),$$

and furthermore that $N(T)$ is a Chebyshev subspace of X, we obtain that $S(T,b)$ is a Chebyshev linear manifold in X. Therefore,

$$\begin{aligned} \operatorname{dist}(\overline{x}, S(T,b)) &= \operatorname{dist}(\overline{T}^M b, T^M b + N(T)) \\ &= \|\overline{T}^M b - T^M b - \pi_{N(T)}(\overline{T}^M b - T^M b)\|, \end{aligned} \tag{3.3}$$

where $\pi_{N(T)}$ is a metric project operator from X into $N(T)$. Since $N(T)$ is a Chebyshev subspace of X, by Theorem 2.6, we obtain that

$$x = \pi_{N(T)}(x) \dotplus C(T), \quad \forall x \in X,$$

where $C(T) = F_X^{-1}(N(T)^{\perp})$. This implies that for all $x_1 \in C(T)$, we have $\pi_{N(T)}(x_1) = 0$. Indeed, the following relation is clearly true

$$x_1 = 0 + x_1, \quad 0 \in N(T),\ x_1 \in C(T).$$

By Theorem 2.6, we have

$$x_1 = \pi_{N(T)}(x_1) + x_2, \quad x_2 \in C(T).$$

Moreover, since the decomposition is unique, we have

$$\pi_{N(T)}(x_1) = 0.$$

Since $N(T) = N(\overline{T})$, then $C(\overline{T}) = C(T)$. Since $\ker \pi_{N(T)}$ is a linear subspace of X, by Lemma 2.7, $\pi_{N(T)}$ is also a linear operator. Since $\overline{T}^M b \in C(\overline{T}) = C(T)$ and $T^M b \in C(T)$, we have

$$\pi_{N(T)}(\overline{T}^M b - T^M b) = \pi_{N(T)}(\overline{T}^M b) - \pi_{N(T)}(T^M b) = 0.$$

By (3.3), there holds

$$\operatorname{dist}(\overline{x}, S(T,b)) = \|\overline{T}^M b - T^M b\|.$$

By (3.2) and $\|\overline{x}\| = \|\overline{T}^M b\|$, we have

$$\|(\overline{T}^M - T^M)b\| \leq \|T^M\|\|\delta T\|\|\overline{T}^M b\|.$$

For all $y \in Y\backslash\{0\}$, there exists a unique $b \in R(\overline{T}) = R(T)$ such that

$$b = \pi_{R(T)}(y) = \pi_{R(\overline{T})}(y).$$

It follows from Theorem 2.13 that

$$\overline{T}^M \pi_{R(\overline{T})}(y) = \overline{T}^M(y), \quad T^M \pi_{R(T)}(y) = T^M(y).$$

Hence,

$$\begin{aligned} \|(\overline{T}^M - T^M)y\| &= \|\overline{T}^M(y) - T^M(y)\| \\ &= \|\overline{T}^M \pi_{R(\overline{T})}(y) - T^M \pi_{R(T)}(y)\| \\ &= \|(\overline{T}^M - T^M)b\| \end{aligned}$$

$$
\begin{aligned}
&\leq \|T^M\|\|\delta T\|\|\overline{T}^M b\| \\
&= \|T^M\|\|\delta T\|\|\overline{T}^M \pi_{R(\overline{T})}(y)\| \\
&= \|T^M\|\|\delta T\|\|\overline{T}^M (y)\| \\
&\leq \|T^M\|\|\delta T\|\|\overline{T}^M\|\|y\|.
\end{aligned}
$$

Therefore,

$$\sup_{\|y\|\neq 0} \frac{\|(\overline{T}^M - T^M)y\|}{\|y\|} \leq \|T^M\|\|\delta T\|\|\overline{T}^M\|$$

and

$$\frac{\|\overline{T}^M - T^M\|}{\|\overline{T}^M\|} \leq \|T^M\|\|\delta T\|.$$

Since $\|T^M\|\|\delta T\| < 1$, we have $0 < 1 - \|T^M\|\|\delta T\| \ < \ 1$ and

$$\frac{\|\overline{T}^M - T^M\|}{\|\overline{T}^M\|} \leq \frac{\|T^M\|\|\delta T\|}{1 - \|T^M\|\|\delta T\|}.$$

Moreover,

$$
\begin{aligned}
\|\overline{T}^M y\| &\leq \|\overline{T}^M y - T^M y\| + \|T^M y\| \\
&= \|(\overline{T}^M - T^M)y\| + \|T^M y\| \\
&\leq \|T^M\|\|\delta T\|\|\overline{T}^M y\| + \|T^M y\|.
\end{aligned}
$$

Therefore,

$$(1 - \|T^M\|\|\delta T\|)\|\overline{T}^M y\| \leq \|T^M y\|,$$

which implies that

$$\|\overline{T}^M y\| \leq \frac{\|T^M\|\|y\|}{1 - \|T^M\|\|\delta T\|},$$

or equivalently

$$\frac{\|\overline{T}^M y\|}{\|y\|} \leq \frac{\|T^M\|}{1 - \|T^M\|\|\delta T\|}.$$

Taking the supremum over $y \in Y\backslash\{0\}$, we have

$$\|\overline{T}^M\| \leq \frac{\|T^M\|}{1 - \|T^M\|\|\delta T\|},$$

and the proof is complete. □

If X and Y are Hilbert spaces, then the Moore–Penrose metric generalized inverses of linear operators between Banach spaces coincide with the Moore–Penrose generalized inverses under usual sense since the metric projector is linear orthogonal projector. It is easy to deduce the following perturbation result from our above result.

Corollary 3.6 *Let X and Y be Hilbert spaces, $T \in B(X, Y)$ be with $\overline{D(T)} = \overline{D(\overline{T})} = X$, and $R(T)$ be a closed subspace of Y. Then there exists the Moore–Penrose generalized inverse T^+*

of T. If $\delta T \in B(X,Y)$, $\|T^+\|\|\delta T\| < 1$, $\overline{T} = T + \delta T$, $R(\delta T) \subset R(T)$ and $N(T) \subset N(\delta T)$, then the Moore–Penrose generalized inverse $\overline{T}^+$ of $\overline{T}$ exists and

$$\|\overline{T}^+\| \leq \frac{\|T^+\|}{1 - \|T^+\|\|\delta T\|}, \quad \frac{\|\overline{T}^+ - T^+\|}{\|\overline{T}^+\|} \leq \frac{\|T^+\|\|\delta T\|}{1 - \|T^+\|\|\delta T\|}.$$

Proof Since $T \in B(X,Y)$ and $R(T)$ is closed, the Moore–Penrose metric generalized inverse T^+ of T exists. Since $\|T^+\|\|\delta T\| < 1$, $R(\delta T) \subset R(T)$ and $N(T) \subset N(\delta T)$, there exists the Moore–Penrose metric generalized inverse $\overline{T}^+$ of $\overline{T}$. By Theorem 3.5, taking $T^M = T^+$ and $\overline{T}^M = \overline{T}^+$, we have

$$\|\overline{T}^+\| \leq \frac{\|T^+\|}{1 - \|T^+\|\|\delta T\|}$$

and

$$\frac{\|\overline{T}^+ - T^+\|}{\|\overline{T}^+\|} \leq \frac{\|T^+\|\|\delta T\|}{1 - \|T^+\|\|\delta T\|}.$$ □

The better result on perturbation of Moore–Penrose generalized inverse in Hilbert space can be found in [25].

Theorem 3.7 *Let $T \in B(X,Y)$ and $N(T)$ be a proximinal subspace of X, $R(T)$ be a Chebyshev subspace of Y. If $\ker \mathcal{P}_{N(T)}$ is a subspace of X and $R(T)$ is approximatively compact, then T has a unique and continuous Moore–Penrose metric generalized inverse T^M.*

Proof By Theorem 3.1, there exists a unique Moore–Penrose metric generalized inverse T^M of T such that

$$T^M y = (I_{D(T)} - \pi_{N(T)})T^- \pi_{R(T)} y, \quad y \in Y.$$

Since $R(T)$ is an approximatively compact Chebyshev subspace of Y, it follows from Lemma 2.10 that $\pi_{R(T)}$ is continuous. Since $I_{D(T)} - \pi_{N(T)}$ and T^- are bounded linear operators, the operator $(I_{D(T)} - \pi_{N(T)})T^- \pi_{R(T)}$ is continuous. Thus, there exists a unique and continuous Moore–Penrose metric generalized inverse T^M of T. □

Lemma 3.8 *Let $T, \delta T \in B(X,Y)$, and $N(T)$ be a proximinal subspace of X, $R(T)$ be a Chebyshev subspace of Y. Assume that $\|T^M\|\|\delta T\| < 1$, $N(T) \subset N(\delta T)$ and $R(\delta T) \subset R(T)$. If $\ker \mathcal{P}_{N(T)}$ is a subspace of X, and $R(T)$ is approximatively compact, then the following results are true:*

(1) *$(I + \delta T T^M) : Y \to Y$ is bounded, invertible and*

$$(I + \delta T T^M)^{-1} = \sum_{k=0}^{\infty} (-1)^k (\delta T T^M)^k, \tag{3.4}$$

where $(I + \delta T T^M)^{-1} \in H(Y)$.

(2) *$\sum_{k=0}^{\infty} (-1)^k (T^M \delta T)^k T^M$ is convergent in $H(Y,X)$ and*

$$\sum_{k=0}^{\infty} (-1)^k (T^M \delta T)^k T^M = T^M (I + \delta T T^M)^{-1}. \tag{3.5}$$

(3) *$(I + T^M \delta T) : X \to F_X^{-1}(N(T)^\perp)$ is bounded, invertible and*

$$(I + T^M \delta T)^{-1} = \sum_{k=0}^{\infty} (-1)^k (T^M \delta T)^k, \tag{3.6}$$

where $(I+T^M\delta T)^{-1}\in B(X,X)$.

(4)

$$T^M(I+\delta TT^M)^{-1}=(I+T^M\delta T)^{-1}T^M. \tag{3.7}$$

Proof (1) Since $N(T)$ and $R(T)$ are Chebyshev subspaces of X and Y, respectively, there exists a unique Moore–Penrose metric generalized inverse T^M of T ($T^M\in H(Y,X)$), where $R(T)$ is a closed set, $D(T^M)=Y$ and $R(T^M)=F_X^{-1}(N(T)^{\perp})$. Since $\|T^M\|\|\delta T\|\le r<1$, δTT^M is quasi-additive on $R(\delta TT^M)\subset R(T)$, it follows from Lemma 3.3 that $(I+\delta TT^M)$ is invertible and

$$(I+\delta TT^M)^{-1}=\sum_{k=0}^{\infty}(-1)^k(\delta TT^M)^k,$$

where $(I+\delta TT^M)^{-1}\in H(Y)$.

(2) Since $\|T^M\|\|\delta T\|\le r<1$, by Corollary 3.2, we have $T^M\delta T\in L(X)$ and

$$\begin{aligned}\|(-1)^k(T^M\delta T)^kT^M\|&=\|(-1)^kT^M(\delta TT^M)^k\|\\&\le\|T^M\|\|\delta TT^M\|^k\\&\le\|T^M\|\,r^k\end{aligned}$$

for all $k=0,1,2,\ldots$. Hence, the series $\sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^kT^M$ is absolutely convergent in $H(Y,X)$. Since $\ker\mathcal{P}_{N(T)}$ is a subspace of X and $R(T)$ is approximatively compact, it follows from Theorem 3.7 that T^M is continuous. By Theorem 3.1, T^M is quasi-additive on $R(T)$. Hence, by $R(\delta T)\subset R(T)$, we deduce that

$$\begin{aligned}T^M(I+\delta TT^M)^{-1}&=T^M\sum_{k=0}^{\infty}(-1)^k(\delta TT^M)^k\\&=\sum_{k=0}^{\infty}T^M(-1)^k(\delta TT^M)^k\\&=\lim_{k\to\infty}[T^M-T^M\delta TT^M+\cdots+(-1)^kT^M(\delta TT^M)^k]\\&=\sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^kT^M.\end{aligned}$$

(3) It is obvious that $\sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^k$ is a bounded operator acting from X to $F_X^{-1}(N(T)^{\perp})$. We claim that

$$(I+T^M\delta T)^{-1}=\sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^k.$$

Indeed, taking arbitrary $x\in X$, we have

$$x=(I-T^MT)x+T^MTx.$$

Since $N(T)\subset N(\delta T)$, thus $\delta T(I-T^MT)=0$. It follows from Corollary 3.2 that $T^M\delta T$ is a bounded linear operator. Hence, by equalities (3.4), (3.5) and the inclusion $N(T)\subset N(\delta T)$, we obtain

$$\left[\sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^k\right](I+T^M\delta T)x$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^k(I-T^MT)x + \left[\sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^k\right](I+T^M\delta T)T^MTx \\
&= (I-T^MT)x + \left[\sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^k\right]T^M(I+\delta TT^M)Tx \\
&= (I-T^MT)x + T^M\left[\sum_{k=0}^{\infty}(-1)^k(\delta TT^M)^k\right](I+\delta TT^M)Tx \\
&= (I-T^MT)x + T^M(I+\delta TT^M)^{-1}(I+\delta TT^M)Tx \\
&= x.
\end{aligned}
$$

We have also

$$x = (I-T^MT)x + T^MTx, \quad x \in F_X^{-1}(N(T)^{\perp}) = R(T^M).$$

Since $\delta T(I-T^MT)=0$, T^M is continuous and quasi-additive on $R(T)$, we have

$$
\begin{aligned}
&(I+T^M\delta T)\left[\sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^k\right]x \\
&\quad= (I+T^M\delta T)\left[x + \sum_{k=1}^{\infty}(-1)^k(T^M\delta T)^kT^MTx\right] \\
&\quad= (I+T^M\delta T)\left[x + T^M\sum_{k=1}^{\infty}(-1)^k(\delta TT^M)^kTx\right] \\
&\quad= (I+T^M\delta T)[x + T^M((I+\delta TT^M)^{-1}Tx - Tx)] \\
&\quad= (I+T^M\delta T)[x + T^M(I+\delta TT^M)^{-1}Tx - T^MTx] \\
&\quad= (I+T^M\delta T)x + (I+T^M\delta T)T^M(I+\delta TT^M)^{-1}Tx - (I+T^M\delta T)T^MTx \\
&\quad= (I+T^M\delta T)x + T^M(I+\delta TT^M)(I+\delta TT^M)^{-1}Tx - (I+T^M\delta T)T^MTx \\
&\quad= (I+T^M\delta T)x + T^MTx - (I+T^M\delta T)T^MTx \\
&\quad= x + T^M\delta T(I-T^MT)x \\
&\quad= x
\end{aligned}
$$

by (3.4) and (3.5). Therefore,

$$(I+T^M\delta T)^{-1} = \sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^k \in B(X).$$

The last statement (4) follows easily from (3.4)–(3.6), and the proof is complete. □

Theorem 3.9 *Let $T \in B(X,Y)$, $\delta T \in B(X,Y)$, and $\overline{T} = T + \delta T$. Assume that $N(T)$ is a proximinal subspace of X, $R(T)$ is a Chebyshev subspace of Y, $\|T^M\|\|\delta T\| < 1$, $N(T) \subset N(\delta T)$ and $R(\delta T) \subset R(T)$. If $\ker \mathcal{P}_{N(T)}$ is a subspace of X and $R(T)$ is approximatively compact, then*

(1) $N(T) = N(\overline{T})$, $R(T) = R(\overline{T})$;

(2) $\overline{T}^M = T^M(I+\delta TT^M)^{-1} = (I+T^M\delta T)^{-1}T^M$;

(3) $\|\overline{T}^M\| \le \frac{\|T^M\|}{1-\|\delta TT^M\|}$;

(4) $\|\overline{T}^M - T^M\| \le \frac{\|T^M\|\|\delta TT^M\|}{1-\|\delta TT^M\|}$.

Proof (1) By Lemma 3.4, we have $N(T) = N(\overline{T})$, $R(T) = R(\overline{T})$.

(2) Since $N(T)$ and $N(\overline{T})$ are Chebyshev subspaces of X, $R(T)$ and $R(\overline{T})$ are Chebyshev subspaces of Y, thus T^M and $\overline{T}^M$ exist. It follows from Lemma 3.8 that the operator $(I + \delta T T^M)$ is invertible and

$$(I + \delta T T^M)^{-1} = \sum_{k=0}^{\infty} (-1)^k (\delta T T^M)^k,$$

where $(I + \delta T T^M)^{-1} \in H(Y)$. Denoting $T^\# := T^M (I + \delta T T^M)^{-1} \in H(Y, X)$, we claim that $T^\# = T^M (I + \delta T T^M)^{-1}$ is the Moore–Penrose metric generalized inverse of $\overline{T}$ and

$$\overline{T}^M = T^M (I + \delta T T^M)^{-1} = (I + T^M \delta T)^{-1} T^M.$$

Indeed,

(i) Since $N(T) \subset N(\delta T)$, then $\delta T (I - T^M T) = 0$. Hence,

$$\begin{aligned}
\overline{T} - \overline{T} T^\# \overline{T} &= [I - \overline{T} T^M (I + \delta T T^M)^{-1}] \overline{T} \\
&= [I - (T + \delta T) T^M (I + \delta T T^M)^{-1}] (T + \delta T) \\
&= [(I + \delta T T^M) - (T + \delta T) T^M] (I + \delta T T^M)^{-1} (T + \delta T) \\
&= (I - T T^M)(I + \delta T T^M)^{-1} (T + \delta T) \\
&= (I - T T^M)(I + \delta T T^M)^{-1} (T + \delta T T^M T + \delta T - \delta T T^M T) \\
&= (I - T T^M)(I + \delta T T^M)^{-1} [(I + \delta T T^M) T + \delta T (I - T^M T)] \\
&= (I - T T^M)(I + \delta T T^M)^{-1} (I + \delta T T^M) T = 0,
\end{aligned}$$

i.e.,

$$\overline{T} = \overline{T} T^\# \overline{T}, \quad \text{on } X.$$

(ii) It follows from (3.7) that

$$T^M (I + \delta T T^M)^{-1} = (I + T^M \delta T)^{-1} T^M.$$

T^M is quasi-additive on $R(T)$, which implies that $T^M (T T^M - I) = 0$ and

$$\begin{aligned}
T^\# \overline{T} T^\# - T^\# &= T^M (I + \delta T T^M)^{-1} \overline{T} T^M (I + \delta T T^M)^{-1} - T^M (I + \delta T T^M)^{-1} \\
&= (I + T^M \delta T)^{-1} T^M \overline{T} T^M (I + \delta T T^M)^{-1} - (I + T^M \delta T)^{-1} T^M.
\end{aligned}$$

Furthermore, $R(\delta T) \subset R(T)$. Thus, $(I + T^M \delta T)$ is a linear operator such that

$$\begin{aligned}
T^\# \overline{T} T^\# - T^\# &= (I + T^M \delta T)^{-1} T^M [(T + \delta T) T^M (I + \delta T T^M)^{-1} - I] \\
&= (I + T^M \delta T)^{-1} T^M [T T^M + \delta T T^M - I - \delta T T^M] (I + \delta T T^M)^{-1} \\
&= (I + T^M \delta T)^{-1} T^M (T T^M - I)(I + \delta T T^M)^{-1} \\
&= 0,
\end{aligned}$$

which means that $T^\# \overline{T} T^\# = T^\#$ on Y.

(iii) Noticing that $N(T) \subset N(\delta T)$, we have $N(T) = N(\overline{T})$ and $\delta T = \delta T T^M T$. Since $T^M T = I - \pi_{N(T)}$, we deduce that

$$\overline{T} = T + \delta T = (I + \delta T T^M) T.$$

Hence,

$$\begin{aligned} T^{\#}\overline{T} &= T^M(I+\delta T T^M)^{-1}(I+\delta T T^M)T \\ &= T^M T = I - \pi_{N(T)} = I - \pi_{N(\overline{T})}. \end{aligned}$$

(iv) It follows from the inclusion $R(\delta T) \subset R(T)$ that $R(\overline{T}) = R(T)$. Hence, $\delta T = T T^M \delta T$. Since $T T^M = \pi_{R(T)}$, we have

$$\overline{T} = T + \delta T = T(I + T^M \delta T),$$

and

$$\begin{aligned} \overline{T} T^{\#} &= T(I+T^M\delta T)(I+T^M\delta T)^{-1}T^M \\ &= T T^M \ = \pi_{R(T)} \ = \pi_{R(\overline{T})}. \end{aligned}$$

Therefore, $T^{\#} = T^M(I+\delta T T^M)^{-1}$ is the Moore–Penrose metric generalized inverse of $\overline{T}$, and

$$\overline{T}^M = T^M(I+\delta T T^M)^{-1} = (I+T^M\delta T)^{-1}T^M.$$

Therefore, we have shown that (2) is valid.

(3) Lemma 3.3 shows that

$$\begin{aligned} \|\overline{T}^M\| &= \|T^M(I+\delta T T^M)^{-1}\| \le \|T^M\|\|(I+\delta T T^M)^{-1}\| \\ &\le \frac{\|T^M\|}{1-\|\delta T T^M\|}. \end{aligned}$$

(4) Lemma 3.3 assures that

$$\begin{aligned} \|\overline{T}^M - T^M\| &= \|(I+T^M\delta T)^{-1}T^M - T^M\| \\ &= \|((I+T^M\delta T)^{-1} - I)T^M\| \\ &\le \|(I+T^M\delta T)^{-1} - I\|\|T^M\| \\ &\le \frac{\|T^M\delta T\|\|T^M\|}{1-\|T^M\delta T\|}. \end{aligned}$$

□

Theorem 3.10 *Let T and δT belong to $B(X,Y)$ and $\overline{T} = T + \delta T$. Assume that $N(T)$ is a proximinal subspace of X, $R(T)$ is a Chebyshev subspace of Y, $\|T^M\|\|\delta T\| < 1$, $N(T) \subset N(\delta T)$ and $R(\delta T) \subset R(T)$. If $\ker \mathcal{P}_{N(T)}$ is a subspace of X, $R(T)$ is approximatively compact and $\overline{y} := y + \delta y \in R(T)$ for all $y, \delta y \in R(T)$, then*

$$\frac{\|\overline{x}-x\|}{\|x\|} \le \frac{\kappa}{1-\kappa\varepsilon_T}\left(\varepsilon_y \frac{\|y\|}{\|T\|\|x\|} + \varepsilon_T\right),$$

where $\kappa = \|T\|\|T^M\|$, $\varepsilon_T = \|\delta T\|/\|T\|$, $\varepsilon_y = \|\delta y\|/\|y\|$, $\overline{x} = \overline{T}^M\overline{y}$ and $x = T^M y$.

Proof Noticing that $\overline{T}^M$ is linear on $R(\overline{T}) = R(T)$, it follows from Theorem 3.9 and Lemma 3.3 that

$$\begin{aligned} \|\overline{x}-x\| &= \|\overline{T}^M\overline{y} - T^M y\| \\ &= \|\overline{T}^M\delta y + (\overline{T}^M - T^M)y\| \\ &= \|\overline{T}^M\delta y + \left[(I+T^M\delta T)^{-1} - I\right]T^M y\| \\ &\le \|\overline{T}^M\|\|\delta y\| + \|(I+T^M\delta T)^{-1} - I\|\|T^M y\| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\|T^M\|}{1-\|T^M\|\|\delta T\|}\|\delta y\| + \frac{\|T^M\|\|\delta T\|\|x\|}{1-\|T^M\|\|\delta T\|} \\
&= \frac{\|T^M\|\|T\|}{1-\|T^M\|\|\delta T\|}\frac{\|\delta y\|}{\|y\|}\frac{\|y\|}{\|T\|} + \frac{\|T^M\|\|T\|\|x\|}{1-\|T^M\|\|\delta T\|}\frac{\|\delta T\|}{\|T\|} \\
&= \frac{\kappa}{1-\kappa\varepsilon_T}\left(\varepsilon_y\frac{\|y\|}{\|T\|} + \varepsilon_T\|x\|\right),
\end{aligned}
$$

which finished the proof. □

Corollary 3.11 *If T satisfies the assumptions of Theorem* 3.10 *and T is surjective, then*

$$
\frac{\|\overline{x}-x\|}{\|x\|} \leq \frac{\kappa}{1-\kappa\varepsilon_T}\left(\varepsilon_y+\varepsilon_T\right),
$$

where $\kappa = \|T\|\|T^M\|$, $\varepsilon_T = \|\delta T\|/\|T\|$, $\varepsilon_y = \|\delta y\|/\|y\|$.

Proof Since T is surjective, for any $y \in Y$, there exists x such that $Tx = y$, i.e., $y - Tx = 0$, and $\|y\| \leq \|T\|\|x\|$. Thus by the proof of Theorem 3.10, we have

$$
\begin{aligned}
\|\overline{x}-x\| &\leq \frac{\kappa}{1-\kappa\varepsilon_T}\left(\varepsilon_y\frac{\|y\|}{\|T\|} + \varepsilon_T\|x\|\right) \\
&\leq \frac{\kappa}{1-\kappa\varepsilon_T}\left(\varepsilon_y+\varepsilon_T\right)\|x\|,
\end{aligned}
$$

which finished the proof. □

When T has linearly generalized inverse T^+, Corollary 3.11 is Proposition 4.2 of [4].

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